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The Local L_1 Saturation Class of the Method of Integrated Meyer–König and Zeller Operators

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1. INTRODUCTION AND MAIN RESULTS

The nth integrated Meyer–König and Zeller operator $\hat{M}_n$, $n \in \mathbf{N}$ (see [2]), associates with a real valued Lebesgue integrable function f defined on $I = [0, 1]$, the function series

$$\hat{M}_n(f, x) = \sum_{k=0}^{\infty} \hat{M}_{nk}(x) \int_{I_k} f(t)\, dt,$$

converging for $0 \leqslant x < 1$, with

$$I_k = \left[\frac{k}{k+n}, \frac{k+1}{k+n+1}\right], \qquad k \in \mathbf{N},$$

and

$$\hat{M}_{nk}(x) = (n+1) \binom{k+n+1}{k} (1-x)^n x^k.$$

$\hat{M}_n(f, \cdot\,)$ can be written as a singular integral of the type

$$\hat{M}_n(f, x) = \int_I H_n(x, t) f(t)\, dt \tag{1.1}$$

with the positive kernel

$$H_n(x, t) = \sum_{k=0}^{\infty} \hat{M}_{nk}(x)\, \chi_k(t),$$

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where χ_k denotes the characteristic function of the interval I_k with respect to I. $\hat{M}_n$ is linear, is positive and satisfies

$$\int_I H_n(x, t)\, dt = 1. \tag{1.2}$$

The sequence $\{\hat{M}_n: n \in \mathbf{N}\}$ generates a linear approximation method on the normed spaces $L_p(I)$, $1 \leqslant p < \infty$, i.e., $\lim_{n\to\infty} \|f - \hat{M}_n(f)\|_p = 0$ for $f \in L_p(I)$.

In [3] it was established that the degree of approximation of this method can be $O(n^{-1})$ for suitable subspaces of $L_p(I)$, $1 < p < \infty$, both globally (i.e., on all of I) and locally (i.e., on sub-intervals of (0, 1)). Moreover, n^{-1} is locally the saturation order of the method $\{\hat{M}_n: n \in \mathbf{N}\}$. The corresponding local saturation class is described in [3, Satz 3.1]. The aim of this paper is to extend these results to the still open case $p = 1$.

We first shall prove the following local direct theorem.

THEOREM 1. *Let* $f \in L_1(I)$, $f' \in BV[a, b]$, $0 < a < b < 1$. *Then for* $[a_1, b_1] \subset (a, b)$ *it holds that*

$$\|f - \hat{M}_n(f)\|_{L_1[a_1, b_1]} = O(n^{-1})(n \to \infty).$$

The proof will be tailored specially for the case of the L_1 norm (where the proof in [3] breaks down) and follows ideas in a paper by Bojanic and Shisha [1].

For $f \in L_1(I)$, $f' \in BV(I)$, the proof simplifies considerably and one obtains the global direct theorem,

$$\|f - \hat{M}_n(f)\|_{L_1(I)} = O(n^{-1})(n \to \infty).$$

The converse to Theorem 1 can be obtained by analogy (with only slight modifications) to the case $p > 1$ [3, Satz 3.1]. Thus, if

$$S_1 = \{f \in L_1(I): f' \in BV[a, b] \text{ and } x(1-x)^2 f'(x) = h(x), x \in [a, b], h \in BV[a, b]\},$$

we have the local saturation theorem.

THEOREM 2. *For* $f \in L_1(I)$ *and* $0 < a < a_1 < b_1 < b < 1$, *there are the following implications*:

(i) $\|f - \hat{M}_n(f)\|_{L_1[a, b]} = O(n^{-1})(n \to \infty)$ *implies* $f \in S_1$;

(ii) $f \in S_1$ *implies* $\|f - \hat{M}_n(f)\|_{L_1[a_1 b_1]} = O(n^{-1})(n \to \infty)$;

(iii) $\|f - \hat{M}_n(f)\|_{L_1[a, b]} = o(n^{-1})(n \to \infty)$ *implies* $f \in S_1$ *with* h *constant*;

(iv) $f \in S_1$ *with* h *constant implies* $\|f - \hat{M}_n(f)\|_{L_1[a_1,b_1]} = o(n^{-1})(n \to \infty)$.

A global saturation theorem for $1 \leqslant p < \infty$ is still missing.

2. Notation and Some Lemmas

Let $\chi_{[a,b]}$ denote the characteristic function of $[a, b] \subseteq [0, 1]$ and let $\lambda_{[a,b]}$ be defined by

$$\lambda_{[a,b]}(t) = 1 - \chi_{[a,b]}(t), \qquad t \in I.$$

The proof of Theorem 1 will be based on the following estimation. For $n \geqslant 2$, $0 \leqslant x \leqslant 1$, and $m \geqslant 1$,

$$\hat{M}_n((t-x)^{2m}, x) \leqslant A_m n^{-m}, \tag{2.1}$$

where A_m is independent of n and x [3, Lemma 2.1]. For $0 \leqslant x \leqslant 1$ and $n \geqslant 1$

$$|\hat{M}_n((t-x), x)| \leqslant A n^{-1}, \tag{2.2}$$

where A is independent of n and x (see [3, (2.3)]). For $g \in L_1(I)$, $[a_1, b_1] \subset (a, b)$, $n \geqslant 1$ and r an arbitrary natural number,

$$\|\hat{M}_n(\lambda_{[a,b]}\, g)\|_{L_1[a_1,b_1]} \leqslant B n^{-r} \|g\|_{L_1(I)}, \tag{2.3}$$

where B is independent of g and n. The proof of (2.3) is similar to the proof of Lemma 2.2 of [3] with some obvious modifications.

3. Proof of Theorem 1

Let $f \in L_1(I)$, $f' \in BV[a, b]$ and $0 < a < a_1 < b_1 < b < 1$.
The partition

$$\hat{M}_n(f) - f = \hat{M}_n(\chi_{[a,b]} f) - \chi_{[a,b]} f + \hat{M}_n(\lambda_{[a,b]} f) - \lambda_{[a,b]} f$$

implies

$$\|\hat{M}_n(f) - f\|_{L_1[a_1,b_1]} \leqslant \|\hat{M}_n(\chi_{[a,b]} f) - \chi_{[a,b]} f\|_{L_1[a_1,b_1]} + o(n^{-1}), \tag{3.1}$$

by (2.3) and the fact that $\lambda_{[a,b]} f = 0$ on $[a_1, b_1]$. Fix $x \in [a_1, b_1]$. From the representation

$$f(t) - f(x) = f'(x)(t-x) + \int_x^t (t-\mu)\, df'(\mu)$$

for t, $x \in [a, b]$, and (1.1) and (1.2) we obtain

$$\begin{aligned}
&\hat{M}_n(\chi_{[a,b]}f, x) - \chi_{[a,b]}f(x) \\
&\quad = \int_a^b H_n(x, t)[f(t) - f(x)]\, dt \\
&\quad = f'(x)\, \hat{M}_n((t-x), x) + \int_a^b H_n(x, t) \int_x^t (t-\mu)\, df'(\mu)\, dt.
\end{aligned}$$

Because of (2.2), the first right-hand term can be majorized by $An^{-1}\|f'\|_{L_1[a,b]}$.

Fix $\delta \in (0, 1)$ and extend f' outside of $[a, b]$ so that $df'(\mu) = 0$ for $\mu \notin [a, b]$. Then

$$\begin{aligned}
&\left| \int_a^b H_n(x, t) \int_x^t (t-\mu)\, df'(\mu)\, dt \right| \\
&\quad \leqslant \int_a^b H_n(x, t)\, |t - x| \cdot \left| \int_x^t |df'(\mu)| \right| dt \\
&\quad \leqslant \int_a^b H_n(x, t)\, |t - x| \int_{-|t-x|}^{|t-x|} |df'(\mu)|\, dt \\
&\quad \leqslant \sum_{j=0}^{[(b-a)/\delta]} I_{nj}(x),
\end{aligned}$$

where

$$I_{nj}(x) = \int_{j\delta \leqslant |t-x| < (j+1)\delta} H_n(x, t)\, |t - x| \int_{-|t-x|}^{|t-x|} |df'(\mu + x)|\, dt.$$

Clearly,

$$I_{nj}(x) \leqslant S_{nj}(\delta, x) \int_{-(j+1)\delta}^{(j+1)\delta} |df'(\mu + x)|,$$

where

$$S_{nj}(\delta, x) = \int_{j\delta \leqslant |t-x| < (j+1)\delta} H_n(x, t)\, |t - x|\, dt.$$

Next we shall estimate the factors $S_{nj}(\delta, x)$ for $j = 0$ and $1 \leqslant j \leqslant [(b-a)/\delta]$ separately. We have

$$\begin{aligned}
S_{n0}(\delta, x) &= \int_{0 \leqslant |t-x| < \delta} H_n(x, t)\, |t - x|\, dt \\
&< \delta \int_I H_n(x, t)\, dt = \delta.
\end{aligned}$$

For $1 \leqslant j$,

$$S_{nj}(\delta, x) \leqslant \frac{1}{(j\delta)^3} \int_{j\delta \leqslant |t-x| < (j+1)\delta} H_n(x, t)(t-x)^4 \, dt$$
$$\leqslant \frac{1}{(j\delta)^3} \hat{M}_n((t-x)^4, x)$$
$$\leqslant \frac{A_2}{(j\delta)^3 n^2}$$

by (2.1).

It follows that

$$|\hat{M}_n(\chi_{[a,b]} f, x) - \chi_{[a,b]} f(x)|$$
$$\leqslant \frac{A}{n} \|f'\|_{L_1[a,b]} + \delta \int_{-\delta}^{\delta} |df'(\mu + x)|$$
$$+ \frac{A_2}{\delta^3 n^2} \sum_{j=1}^{[(b-a)/\delta]} \frac{1}{j^3} \int_{-(j+1)\delta}^{(j+1)\delta} |df'(\mu + x)|.$$

Integrating this last inequality with respect to x, and taking into account that

$$\int_{a_1}^{b_1} \int_{-(j+1)\delta}^{(j+1)\delta} |df'(\mu + x)| \, dx \leqslant 2(j+1)\delta \, \|f'\|_{BV[a,b]},$$

we obtain

$$\|\hat{M}_n(\chi_{[a,b]} f) - \chi_{[a,b]} f\|_{L_1[a_1,b_1]}$$
$$\leqslant \frac{A(b_1 - a_1)}{n} \|f'\|_{L_1[a,b]} + 2\delta^2 + \frac{2A_2}{\delta^2 n^2} \|f'\|_{BV[a,b]} \sum_{j=1}^{\infty} \frac{j+1}{j^3}$$
$$= O(n^{-1}), \tag{3.2}$$

if we choose $\delta = n^{-1/2}$.

Combining (3.1) and (3.2) completes the proof.

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